**7. Symmetries and Conservation Laws**

**7.1 Preliminaries**

In modern physics, Conservation laws are really important.

Suppose two particles are traveling in a straight line and they have the same mass, and their positions are q1 and q2 respectively.

Their Langrangian is given as

Potential Energy is a function of a combination of particles. In our case,

Let's denote the Time derivative of the Potential Energy

As

So the laws of motion are

Now let's see if momentum is conserved!

So let's add them together.

Now we said that the Potential is a combination of q1 and q2. So lets set it as a linear combination

So now the laws of motion are:

So there is our conservation law:

(bp1+ap2 )=0

**7.2 Examples of Symmetries**

**Passive change:** A **change** that **doesn’t affect** the system.

**Active change:** A **change** that **moves** the system to a **new configuration-space.**

**Symmetry: Symmetry** is a **coordinate transformation,** that **doesn’t affect the Langrangian.**

Let's consider a shift in the coordinates of a particle in a closed system from qi to q’i.

q’i = q’i(qi)

Now let’s consider a system with 1 degree of freedom with a Langrangian

Now suppose we make a change in q by shifting it by a small amount δ

δ

q+δ

q

Let’s assume that δ is simply a constant and does depend on time. If that is the case, then that means that velocity doesn’t change. It is equal to the velocity that we would have at that time if we didn’t shift our coordinates. Now since velocity doesn’t change, the Langrangian doesn’t change either!

So δL = 0

**Translation Symmetry:** Translation symmetry is called the symmetry of a **system that undergoes a shift in coordinates in space, that is described by adding a constant to the coordinate system.**

You might have seen this coming, but what this essentially means, is that whenever there is a Potential Energy in the system, after such a transformation, the Langrangian changes too.

**When the Langrangian doesn’t change when there is a shift in the coordinate system, we say that it is invariant over change.**

If the change of coordinate system Is simply a rotation, it follows that it doesn’t affect the Langrangian.

We also say that δ is an infinitesimal change

**Continuous transformation:** Transformations which by repeating a process, build up a finite change.

**Infinitesimal Transformation:**  A small shift of the coordinate system that depends on the value for the coordinates. δqi = fi(q)δ

**Continuous Symmetry:** A continuous symmetry is an infinitesimal transformation of the coordinates for which the Langrangian doesn’t change.

**7.3 Consequences of Symmetry**

(Hard Part) Optional

Now let’s see what happens when we try to compute the change of the Langrangian when we shift qi and by **δ.**

So

Since we have shown earlier that is the momentum conjugate to qi , (pi) and since we assume that the system evolves along the trajectory that satisfies the Euler-Langrange Equations we set

So now we can rewrite the initial equations as

Now we can use the product rule from Calculus

(BUT IN REVERSE)

So we see that

And since we said that

δqi = fi(q)δ we see that

Which is nothing but the proof for the Conservation Law

**7.5 Back to examples**

**For any system of particles, if the Langrangian is invariant over change under simultaneous translation of the positions of all particles, then momentum, is conserved.**

And so Newtons 3rd law can be translated to :

**Nothing in the Laws of physics changes if everything is simultaneously shifted in space.**

**Angular Momentum:** A quantity that involves both coordinates and momenta:

l = ypx – xpy

**For any system of particles, if the Langrangian is invariant over change under simultaneous rotation of the positions of all particles, about the origin, angular momentum is conserved.**

Exercise: Very Hard Optional Try to find the Laws of motion of a double pendulum :

The masses of the objects are both 1kg and the length of the ropes are both 1 m